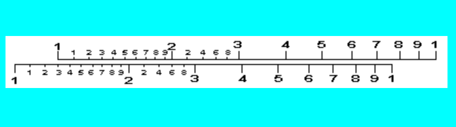
Use of the Slide Rule as a Teaching Tool



1. Introduction

In the days before calculators and personal computers an engineer always had a slide rule nearby. These days it is difficult to locate a slide rule outside of a museum. I don't know how many of those who have used a slide rule ever thought of it as an analog computer, but that is really what it is. As such, I think that it is an ideal tool for teaching the mathematical concept of transformations. With the proper scales, a slide rule can be used not only to multiply but to find the third side of a right triangle and add velocities relativistically. In order to introduce the concepts that will be used, we will start with the simplest type of slide rule - one that is made up of two ordinary rulers.

2. Using Rulers for Addition

Two rulers can be lined up as in the figure below to show that 5 + 3 = 8. Note that in order to do this it is not necessary to know how to add or even to be able to count. All that is required is to line up the start of the top ruler with the symbol "5" on the bottom ruler; locate the symbol "3" on the top ruler and the symbol "8" below it on the bottom ruler.

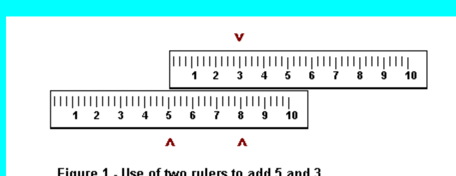


Figure 1 - **Use of two rulers to add 5 and 3**

For the purposes here I am going to define a slide rule as using two copies of the same scale, with one scale moving relative to the other. Actual slide rules may use two different scales, which could either be stationary or mobile with respect to the other.

3. Analog Computers

An analog computer performs a calculation by transforming numbers into physical quantities, combining these physical quantities and then converting the result into the result of the calculation. The way in which a slide rule is an analog computer should be clear - a numerical computation (in this case addition) is performed by changing numbers into a physical representation (distances); the distances are added and the resultant distance is reverse transformed into the result of the numerical calculation.

4. Mirror Worlds

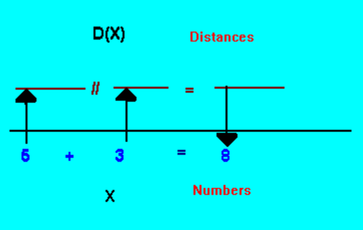


Figure 2 - Converting numbers to distances

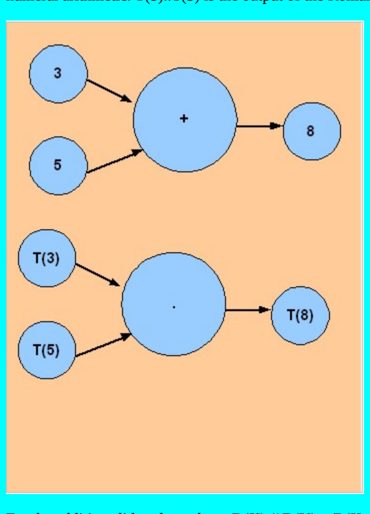
Figure 2 above shows diagrammatically what is happening. Imagine two worlds, a number world X below the black horizontal line and a distance world above it. In the number world 3 is added to 5 to get 8. The arrows above "5" and "3" represent the transform of these numbers into distances. The symbol "//" represents the operation of adding the two distances to produce the distance to the right of the "//". The arrow above the "8" represents the transform of that distance into the number 8. There are thus two ways to get to the "8". Symbolically, 5 + 3 = D^{-1}(D(5) // D(3)). That is, adding 5 and 3 is the same as adding their distances and then taking the inverse transform to convert the distance 8 into the number 8.

5. Translations

The addition slide rule was compared above to a mirror. Perhaps a more apt metaphor would be a translation.

Suppose you traveled back in time to the days of the Roman Empire. You notice someone doing arithmetic using Roman numerals and you want to verify your understanding of this numeric representation. Let T be the transformation from Arabic numerals to Roman numerals. According to what you were taught T(5) = V, T(III) = 3 and T(8) = VIII.

You observe that this particular person seems to be using ".." to stand for "+". You hand over a sheet of paper with V .. III written on it. When the person writes VIII on the paper this increases your confidence that the above transformation is correct. You have established a degree of internal consistency. We can think of addition as taking two input values to produce an output value. Since the output is determined by the inputs, for one process to be a translation of the other it is sufficient for equivalent inputs to result in equivalent outputs. (5+3) is the output of the Arabic numeral arithmetic. T(5)..T(3) is the output of the Roman numeral arithmetic. For the outputs to be equivalent we must have T(5)..T(3) = T(5+3) . If we could prove that in general T(X)..T(Y) = T(X+Y)then we could say that the one process is a translation of the other.



For the addition slide rule we have D(X) // D(Y) = D(X+Y) . The combining of distances is a translation of the addition of numbers. It does not matter that numbers and distances are different entities. Adding numbers is structually equivalent to combining distances. The next section states this in more formal terms.

6. Isomorphisms

Any invertible relationship of the form
T(x & y) = T(x) % T(y)
is called an isomorphism (Greek for "same form").
The & and % stand for operators on the elements of the appropriate domain. In the above example & = + and % = //.

The inverse transform T^{-1} provides a way of reversing the roles of the mirror domain and the original domain or, alternatively, of reversing the direction of the translation.

Let u=T(x), v = T(y).

Then x = T^{-1}(u), y = T^{-1}(v).
Substituting in the above equation,
T(T^{-1}(u) & T^{-1}(v)) = u % v.

Applying T^{-1} to both sides,
T^{-1}(u % v) = T^{-1}(u) & T^{-1}(v).

In an isomorphism variables in one domain are related to each other through & operator in exactly the same way as the transformed variables relate to each other through the % operator.

In particular, one operator will be commutative if and only if the other is:

x & y = y & x if and only if
T(x) % T(y) = T(y) % T(x).

This is easy to show:
If y & x = x & y then
T(y) % T(x) = T(y & x) = T(x & y) = T(x) % T(y).

All operations for which there are slide rule scales are commutative since adding distances is commutative.

7. A Little Philosophy

Equations used to express scientific laws are an expression of an isomorphism between nature and mathematics. Analog computers are a reversal of the usual computation. Ordinarily, the isomorphism is used to determine a physical quantity by measuring the other physical quantities in the equation and then using the equation to solve for the missing quantity. In an analog computer, a number is computed by converting the other numbers in the equation into physical quantities and then using the physical situation to determine the missing value.

We use isomorphisms in our daily lives all the time. When we look in a mirror or read a map we are using isomorphisms. Whenever we solve a problem by an analogy to another problem there is an implied isomorphism.

In mathematics, isomorphisms are used to express relationships between abstract mathematical objects. The slide rule can be used by the teacher to both teach the concept of isomorphism and to unify, and thus simplify, several different concepts by showing how they are examples of isomorphisms.

I am going to present logarithms as an isomorphism between multiplication and addition, without reference to exponents. If you think that this presentation is unnatural, consider that this is the point of view taken by the discoverer of logarithms, John Napier, in the seventeenth century. He was unaware of the connection between logarithms and exponents until it was brought to his attention. Napier was just looking for a simpler way of multiplying.

8. Logarithms, Multiplication and Composition

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 The logarithmic function satisfies the relationship
log (x * y) = log(x) + log(y).

 The log function is an isomorphism. It transforms a
multiplication problem into an addition problem. Since
addition of numbers is isomorphic to addition of
distances, consider the effect of applying the distance
function D to both sides of the equation.

D(log(x*y)) = D( log(x) + log(y) ) = D( log(x)) // D(log(y) )
D(log(x*y)) = D(log(x)) // D(log(y))

It follows that the transform (D log) formed by composing
the D and log transforms is an isomorphism. The same
process could be used to show that in general the
composition of two isomorphic transforms is an isomorphic transform -
 the mirroring of the mirroring of a domain is itself a
mirroring of the domain;
the translation of a translation is a translation of the original.

 The construction of a slide rule for multiplication
follows from the above equation. If the distance
that a number is placed is equal to the log of the
number then the result is the standard slide rule.
Figure 3 shows how the slide rule is used to
multiply 10 and 1000.
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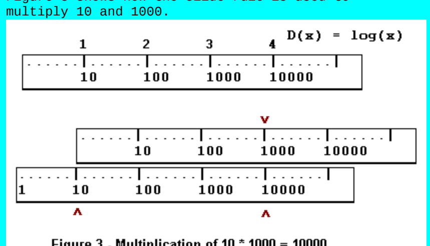


Figure 3 - **Multiplication of 10 * 1000 = 10000**

Shortly after their discovery it was realized that logarithm functions $\log_a(x)$ were the inverse of exponential functions a^x. For convenience, let us write the exponential function as EXP(X) and the logaritm function as LOG(X) for some common base a.

Since LOG(X) is an isomorphism and EXP(X) is its inverse, then by what was shown above we get:

EXP(X+Y) = EXP(X) * EXP(Y),
which is an expression of the law of exponents.

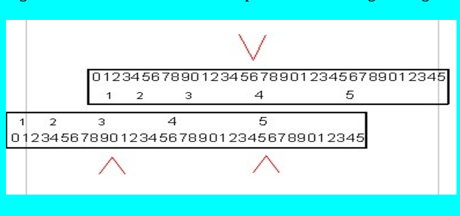
9. Right Triangles and Parallel Resistances

What made it possible to construct a slide rule for multiplication was that the log function provides an isomorphism between multiplication and addition. There are other isomorphisms to addition. Consider right triangles. The length of the hypotenuse is given by:

$c^2 = a^2 + b^2$.

If we let $s(x) = x^2$ and define a & b as the length of the hypotenuse of a triangle with sides of lengths a and b then s(a & b) = s(a) + s(b) and we have our isomorphism. If a slide rule is constructed so that the distance along the slide rule is equal to the square of the number then the slide rule can be used to find the third side of a right triangle given the other two.

Figure 4 shows such a slide rule set up for the familiar right triangle with sides of 3, 4 and 5.



A Right Triangle Slide Rule

As another example consider the equivalent electrical resistance for two resistors in parallel given by $1/r = 1/r_1 + 1/r_2$. This can be computed using a slide rule with distances equal to the reciprocal of the number.
Here is a parallel resistor slide rule.
A parallel resistor slide rule.
Note that in this case the numbers decrease in going from left to right. The left edge of the slide rule is at the "i" on the left side, representing infinite resistance.

10. Relativistic Velocity Addition

When a car traveling with a velocity of u approaches a car traveling with velocity v, the velocity w with which they pass each other is very close to u+v. This formua is not exact and for very high velocities will be in error. The proper relationship between u, v and w satisfies the formula: (c+w)/(c-w) = (c+u)/(c-u) * (c+v)/ (c-v), where c is the speed of light (about 186,000 miles per second).

To avoid having to use the speed of light in our calculations we can express all the velocities as fractions of the speed of light. Dividing the numerator and denominator of all three terms by c gives :
(1 + p)/ (1 - p) = (1 + q)/ (1 - q) * (1 + r)/ (1 - r), where
p = w/c, q=u/c and r = v/c.

Applying in succession the log and distance functions to both sides gives:
D (log ((1 + p)/ (1 - p))) =
D (log ((1 + q)/ (1 - q))) // D (log ((1 + r)/ (1 - r))).

To construct a slide rule to add velocities, set the distance of a fraction x equal to
log ((1+x) / (1-x)).
Figure 5 shows a slide rule for adding velocities.

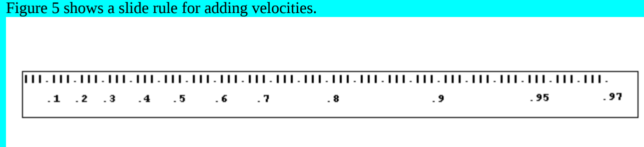


Figure 5 - **Slide rule for adding velocities**

A Relativistic Velocity Addition Slide Rule

11. Proof by Isomorphism

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 Consider again the original formula for velocity
(c + w) / (c - w) = (c + u)/(c - u) * (c + v)/(c -v).

Let u & v be the sum of velocities u and v.
Let T(x) = (c + x)/(c - x).
Then T(u & v) = T(u)*T(v)

How do we add three velocities? In the example
of the person walking with velocity u in a
train traveling with velocity v, let s be the velocity
of the earth relative to the sun. What is the
velocity of the person relative to the sun?

 Velocities can only be added two at a time. There
are two ways of doing this and we would hope that
they come out the same.

 We could first find the velocity of the person
relative to the earth and then add this velocity to
 the velocity of the earth relative to the sun.
 This would give

T(s & (v & u)) = T(s) * T(v & u) = T(s) * (T(v) * T(u))

On the other hand we could first find the velocity of
the train relative to the sun and then add the
person's velocity. We then have

T((s & v) & u) = T(s & v) * T(u) = (T(s) * T(v)) * T(u).

The two values are of course the same. The reason
for this is that multiplication is associative, i. e.,
(a * b) * c = a * (b * c). We see that this causes
addition of velocities to be associative -
s & (u &v) = (s & u) &v.

 The above argument could have been used for
any isomorphism. Thus we have the property
that isomorphisms preserve the associative
property just we showed earlier that they preserve
the commutative property. We could have used
this property to state immediately that it does
not matter which of the two ways the velocities
are added because multiplication is associative.

 It could also have been argued that using the
velocity slide rule, it is obvious that it does not
make any difference in which order the velocities
are added. This is because the addition of distances
is both commutative and associative and any
calculation for which we can construct a slide
rule must therefore also be both commutative and
associative.

 In the above formula for velocity addition it is
possible to solve for w to get

u & v = (u + v)/ (1 + u*v/c^2).
To show that addition of velocities is associative
we could have then solved explicitly for both
(s & (v & u)) and ((s & v) & u), but this involves
a great deal more effort.



We can also apply the above results to the parallel resistor and
right triangle examples.

Using the notation in the section on right triangles, the
distance (x & y) that results from East and North displacements
of x and y is given by s(x & y) = s(x) + s(y). To generalize to
three dimensions we get s(s & y & z) = s(x) + s(y) + s(z).

For combining several parallel
resistors we get:

1/r = 1/r_1 + 1/r_2 + ... + 1/r_n.
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Boolean Algebra

I am going to present an example of a non-numeric isomorphism. Unfortunately, there is no corresponding slide rule.

Although only a small portion of high school students is likely to become computer programmers, most of them will probably be using computers in one way or another. The distinction between program developers and program users has been blurred by such software as spreadsheets and database query programs. There are thus good practical reasons for introducing Boolean algebra in high school. The DeMorgan rules reveal a fundamental isomorphism that facilitates teaching this subject and makes it more interesting.

The variables in Boolean algebra can take on only two values - TRUE or FALSE and the principal operations are AND, OR and NOT. The variables can stand for any statement; we are concerned with whether the combination of such statements is either true or false.

The AND, OR and NOT operators agree with what common sense would say they should be. NOT is the simplest operator. It takes only one variable.
NOT X = FALSE if X is TRUE and TRUE if X is FALSE.

Because Boolean variables, unlike numerical values, take on only two values we can completely specify the AND and OR operations with tables.

X AND Y

		Y	
		TRUE	**FALSE**
X	**TRUE**	*TRUE*	*FALSE*
	FALSE	*FALSE*	*FALSE*

Whenever possible, check to see if a result "makes sense". **The above table says that X AND Y is TRUE only if both X and Y are both TRUE**, in agreement with how the term AND is used in everyday use.

X OR Y

		Y	
		TRUE	**FALSE**
X	**TRUE**	*TRUE*	*TRUE*
	FALSE	*TRUE*	*FALSE*

The table for X OR Y says that X OR Y is FALSE only if both X and Y are both FALSE.

One complication is that the word "or" in English can have two different meanings; which one is being used can usually be determined from the context of the statement. "or" can be defined as above or it can be the same except that it is defined as FALSE when X and Y are both TRUE. For example, if I say "Either candidate A or candidate B will be the next president", it is understood that this rules out the possibility of both A and B being the next president. This type of "or" is referred to in logic as an exclusive or (XOR) and by way of contrast OR is sometimes referred to as an inclusive or. In this section I will only be dealing with the inclusive or.

Notice that the summary of the X OR Y table is the same as the summary of the X AND Y table with the words TRUE and FALSE interchanged. This suggests the following isomorphism, which is one of DeMorgan's two rules:

NOT (X AND Y) = (NOT X) OR (NOT Y)

In plain language this says that if the statement X AND Y is not TRUE then either X is FALSE or Y is FALSE. We can formally prove the statement by using the two above tables to show that both sides of the equation are equal for all four combinations of values for X and Y.

To get the other of DeMorgan's rules we apply the inverse of NOT to show the isomorphism in the other direction. NOT is its own inverse so we get:

NOT (X OR Y) = (NOT X) AND (NOT Y)

AND and OR are isomorphic. We could in principle discard OR from our vocabulary and just use AND, though I would not recommend this unless you are planning a career in politics. "It will rain today or tomorrow" would become "It is not true that it will not rain today and it will not rain tomorrow". There have been times, however, when I have used DeMorgan's rules when programming to simplify statements.

The isomorphism simplifies proofs. We can use the table to show that AND is commutative and associative. It follows immediately by isomorphism that the same is true of OR.

From Isomorphism to Duality

We can go further. The fact that NOT is the transform function for both AND and OR means that we can apply both transforms simultaneously. For example, consider the following distributive identity between AND and OR which is analogous to the distributive operation of multiplication and addition:

X AND (Y OR Z) = (X AND Y) OR (X AND Z).

We can prove this statement by substituting all 8 combinations of X, Y and Z. We should also test the reasonableness of the statement by using an example.
"I will speak to Sarah and (I will speak to) John or Raymond." is the same as "I will speak to Sarah and John or I will speak to Sarah and Raymond."

Apply NOT to both sides of the equation. Using the AND isomorphism gives the following on the left side.

NOT(X AND (Y OR Z)) = NOT X OR NOT (Y OR Z)

Applying the OR isomorphism gives

NOT X OR NOT (Y OR Z) = NOT X OR ((NOT Y) AND (NOT Z))

Each of the arguments has been negated and the ANDs and ORs have been interchanged. The same happens on the right side of the equation. We get

NOT X OR ((NOT Y) AND (NOT Z)) =
((NOT X) OR (NOT Y)) AND ((NOT X) OR NOT Z)

We can get rid of the NOTs by setting X'=NOT X, Y'=NOT Y, Z'=NOT Z, so that what end up is the same form as we started except the ANDs and ORs have been interchanged:

X' OR (Y' AND Z') = (X' OR Y') AND (X' OR Z')

For every identity a new one can be created by interchanging AND and OR.

The relationship between AND and OR is referred to as a duality. In this case we have a duality between operators. There are different types of duality but the general principle is that a dualism exists when true statements can be generated from other true statements by interchanging two terms.

Addendum - Slide Rule Scales for Raising a Number to a Power

Since, in general, x^y is not equal to y^x, there can not be slide rule scales for raising a number to a power if we require both scales to be the same. However, if we remove the requirement for isomorphism we can use two different scales to achieve our purpose. We can create the slide rule scales if we can find two different functions T and U that satisfy:

$T(x^y) = T(x) + U(y)$

We then create the slide rule having T(x) as the bottom scale and U(y) as the top scale. Standard slide rules do this by having T(x) = log(log(x)) and U(y) = log(y).

To see why this works, find $\log(\log(x^y))$:

$\log(\log(x^y)) = \log(y*\log(x)) = \log(y) + \log(\log(x))$.

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